

# Linear evolution equations in scales of Banach spaces

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**Abstract:** We study linear evolution equations in a scale of Banach spaces, which have, by construction, the so-called worsening property. Existence of classical solutions, weak uniqueness and continuous dependence on the generator and initial condition is shown. The worsening property leads to new existence and uniqueness results which cannot be obtained by the well-developed semigroup methods. The results are applied to an infinite system of ordinary differential equations. In particular, it is shown that the solutions can be approximated by solutions to certain finite dimensional systems of ordinary differential equations. We provide weak uniqueness for solutions to Fokker-Planck equations related with Markov evolutions in the continuum.

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## 1 Introduction

Linear evolution equations in Banach spaces became an important part of classical and applied mathematics, cf. [Kat70, Kat73, Paz83, EN00] and the references therein. However, many equations motivated by applications do not admit solutions evolving in one fixed Banach space. Such solutions rather belong to an inductive limit of Banach spaces. Particular classes of inductive limits of Banach spaces can also be described by a suitable scale of Banach spaces, cf. [Bea72]. The solution then has, by construction, the so-called worsening property, cf. [Ovs74, Ovs80, Ovs13]. Applications to certain partial differential equations are considered, e.g., in [Saf95, Hen13, BHP15]. In the last years we also observe a growing interest to Fokker-Planck equations related with Markov evolutions in the continuum, cf. [FKO12, FK13, KK16]. Common techniques typically provide existence and uniqueness of classical solutions in an appropriately chosen scale of Banach spaces.

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However, the notion of classical solutions to Fokker-Planck equations on infinite dimensional spaces is only of limited use. A wider and more natural framework is given by the notion of weak solutions to Fokker-Planck equations, cf. [WZ02, WZ06, Lem10, RZ10]. The aim of this work is to provide conditions under which one can construct an evolution system having the worsening property and weak uniqueness for the associated evolution equations holds. As a consequence, we obtain weak uniqueness for the Fokker-Planck equations considered, e.g., in [BKKK13, FKKZ14, Fin15, FKKO15, KK16].

Let  $\mathbb{B} = (\mathbb{B}_\alpha)_{\alpha > \alpha_*}$  be a scale of Banach spaces with  $\alpha_* \in \mathbb{R}$  and

$$\mathbb{B}_{\alpha'} \subset \mathbb{B}_\alpha, \quad \|\cdot\|_\alpha \leq \|\cdot\|_{\alpha'}, \quad \alpha' < \alpha. \quad (1.1)$$

Denote by  $i_{\alpha'\alpha} \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$  the corresponding embedding operator. Here and in the following  $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$  stands for the space of all bounded linear operators from  $\mathbb{B}_{\alpha'}$  to  $\mathbb{B}_\alpha$ . Denote by  $\|\cdot\|_{L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)}$  its operator norm. We also use the notation  $\|\cdot\|_{\alpha'\alpha}$  if no confusion can arise. Let  $x = y$ ,  $x \in \mathbb{B}_{\alpha'}$ ,  $y \in \mathbb{B}_\alpha$  stand for  $i_{\alpha'\alpha}x = y$ . A bounded linear operator  $L$  in the scale  $\mathbb{B}$  is, by definition, a collection of bounded linear operators from  $\mathbb{B}_{\alpha'}$  to  $\mathbb{B}_\alpha$ , i.e.  $L = (L_{\alpha'\alpha})_{\alpha' < \alpha}$ ,  $L_{\alpha'\alpha} \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ , satisfying for  $\alpha' < \alpha < \alpha''$

$$L_{\alpha'\alpha''} = i_{\alpha\alpha''} L_{\alpha'\alpha} = L_{\alpha\alpha''} i_{\alpha'\alpha}. \quad (1.2)$$

By  $L \in L(\mathbb{B})$  we indicate that  $L$  is a bounded linear operator in the scale  $\mathbb{B}$ . In the following we omit the subscripts  $\alpha'\alpha$  when no confusion can arise. Abusing notation we let  $\mathbb{B} := \bigcup_{\alpha > \alpha_*} \mathbb{B}_\alpha$ , then  $\mathbb{B}$  is a vector space and any operator  $L \in L(\mathbb{B})$  is represented by a linear map  $L : \mathbb{B} \longrightarrow \mathbb{B}$  such that its restrictions  $L_{\alpha'\alpha} : \mathbb{B}_{\alpha'} \longrightarrow \mathbb{B}_\alpha$  are bounded linear operators.

Let  $(L^\Delta(t))_{t \geq 0} \subset L(\mathbb{B})$  be continuous in  $t$  w.r.t. the strong topology, that is  $t \longmapsto L^\Delta(t)k \in \mathbb{B}_\alpha$  is continuous for any  $k \in \mathbb{B}_{\alpha'}$  and  $\alpha' < \alpha$ .

**Definition 1.1.** *A local (forward) worsening evolution system on  $\mathbb{B}$  associated to  $L^\Delta(t)$  is a family of linear operators  $(W^\Delta(t, s))_{0 \leq t-s < T(\alpha', \alpha)}$  on  $\mathbb{B}$ , where  $T(\alpha', \alpha) := \frac{\alpha - \alpha'}{M(\alpha)}$  and  $M(\alpha) \geq 0$  is continuous and increasing. This family satisfies  $W^\Delta(t, s) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$  for all  $\alpha' < \alpha$ ,  $(t, s) \longmapsto W^\Delta(t, s)k \in \mathbb{B}_\alpha$  is continuously differentiable with derivatives*

$$\frac{\partial}{\partial t} W^\Delta(t, s)k = L^\Delta(t)W^\Delta(t, s)k \quad (1.3)$$

$$\frac{\partial}{\partial s} W^\Delta(t, s)k = -W^\Delta(t, s)L^\Delta(s)k \quad (1.4)$$

for any  $0 \leq t - s < T(\alpha', \alpha)$  and  $k \in \mathbb{B}_{\alpha'}$ . For  $\alpha' < \alpha$  and  $s \leq t$  let

$$\alpha(t, s, \alpha') := \inf \{ \beta \geq \alpha' \mid W^\Delta(t, s)k \in \mathbb{B}_\beta \}.$$

It is also assumed to satisfy

$$W^\Delta(t, s)k = W^\Delta(t, r)W^\Delta(r, s)k, \quad k \in \mathbb{B}_{\alpha'} \quad (1.5)$$

for  $s \leq r \leq t$ ,  $0 \leq s \leq r < s + T(\alpha', \alpha)$  and  $t < \min\{s + T(\alpha', \alpha), r + T(\alpha(r, s, \alpha'), \alpha)\}$ . We call  $W^\Delta(t, s)$  a global worsening evolution system if  $M(\alpha)$  is bounded. The evolution system preserves regularity if  $M(\alpha) = 0$ , that is  $T(\alpha', \alpha) = \infty$ .

The operator  $L^\Delta(t)$  is also called generator of  $W^\Delta(t, s)$ . If  $M(\alpha) \leq M$  for some  $M$ , then  $\frac{\alpha - \alpha'}{M} \leq T(\alpha', \alpha)$  and hence one can extend  $W^\Delta(t, s)$  to all  $0 \leq s \leq t$ . Note that a regularity preserving evolution system belongs, by definition, to  $L(\mathbb{B})$ .

**Remark 1.2.** The function  $\alpha(t, s, \alpha')$  satisfies  $\alpha(s, s, \alpha') = \alpha'$ ,  $\alpha(t, s, \alpha') \leq \alpha$  and since  $T(\alpha', \alpha)$  is continuous also  $\alpha(t, s, \alpha') \leq \beta < \alpha$  for some  $\beta \in (\alpha', \alpha)$ . In particular, (1.5) makes sense.

**Remark 1.3.** All considerations can be adapted to the case of an backward local worsening evolution system and will be later on used without any further reference.

We provide sufficient conditions for the construction of local worsening evolution systems. In particular, existence and uniqueness of classical solutions to the forward equation

$$\frac{\partial u(t)}{\partial t} = L^\Delta(t)u(t), \quad u(s) = k \in \mathbb{B}_{\alpha'}, \quad t \geq s \quad (1.6)$$

and to the backward equation, where  $T^* > 0$ ,

$$\frac{\partial v(s)}{\partial s} = -L^\Delta(s)v(s), \quad v(T^*) = k \in \mathbb{B}_{\alpha'}, \quad s \in [0, T^*) \quad (1.7)$$

is shown. The construction is used to show that the solutions depend continuously on the generator  $L^\Delta(t)$  and on the initial condition. Concerning weak uniqueness let  $\mathbb{E} = (\mathbb{E}_\alpha, \|\cdot\|_\alpha)_{\alpha > \alpha_*}$  be a scale of Banach spaces with

$$\mathbb{E}_\alpha \subset \mathbb{E}_{\alpha'}, \quad \|\cdot\|_{\alpha'} \leq \|\cdot\|_\alpha, \quad \alpha' < \alpha. \quad (1.8)$$

Suppose that  $\mathbb{B}_\alpha = \mathbb{E}_\alpha^*$  and assume that there exists a family of operators  $L(t)$  on  $\mathbb{E}$  such that  $L(t)^* = L^\Delta(t)$ . Let  $\mathcal{Y} \subset \bigcap_{\alpha > \alpha_*} \mathbb{E}_\alpha$  be a dense subspace for each  $\mathbb{E}_\alpha$ . We give sufficient conditions for weak uniqueness of solutions to

$$\frac{d}{dt} \langle G, u(t) \rangle = \langle L(t)G, u(t) \rangle, \quad u(s) = k, \quad G \in \mathcal{Y} \quad (1.9)$$

and for  $T^* > 0$  to

$$\frac{d}{ds} \langle G, v(s) \rangle = -\langle L(s)G, v(s) \rangle, \quad v(T^*) = k, \quad G \in \mathcal{Y}. \quad (1.10)$$

The results are applied to the system of ordinary differential equations

$$\frac{du_n(t)}{dt} = \sum_{k=0}^{\infty} a_{nk} u_k(t), \quad u_n(0) = x_n, \quad n \in \mathbb{N}_0, \quad (1.11)$$

where  $a_{nk}$  are complex numbers and  $u_n : \mathbb{R}_+ \longrightarrow \mathbb{C}$  are continuous differentiable. Such equations are well-studied by semigroup methods if  $(a_{nk})_{n,k=0}^\infty$  is a Kolmogorov matrix, cf. [BA06, BLM06, TV06]. However, the study of spatial birth-and-death dynamics leads to a system of Banach space valued differential equations for which semigroup methods do not provide satisfactory results. Nevertheless, we will show that such equations can be solved in a suitable chosen scale of Banach spaces. In particular, (1.11) has exactly one component-wise solution  $(u_n(t))_{n=0}^\infty$  and this solution depends continuously on  $a_{nk}$ . The extension to systems of Banach space valued differential equations is considered in the last section.

This work is organized as follows. Section two is devoted to the study of regularity preserving forward and backward evolution systems on the scale  $\mathbb{B}$ . The third and main section is devoted to the construction and properties of local worsening evolution systems. We prove existence and uniqueness of classical solutions to (1.6) and (1.7), see Theorem 3.1 and Theorem 3.7. Afterwards we study stability and prove a certain monotonicity property in ordered Banach spaces, see Theorem 3.3 and Theorem 3.8. Weak uniqueness for (1.9) and (1.10) is proved in Theorem 3.10. Section four shows how this results can be applied to (1.11), whereas weak uniqueness of solutions to the Fokker-Planck equation considered in [KK16] is established in the last section.

## 2 Regularity preserving evolution systems

Let  $\mathbb{B}$  be a scale of Banach spaces with property (1.1). Given two operators  $L, K \in L(\mathbb{B})$ , the composition  $LK \in L(\mathbb{B})$  is defined by

$$(LK)_{\alpha'\alpha} := L_{\beta\alpha} K_{\alpha'\beta}, \quad (2.1)$$

where  $\beta \in (\alpha', \alpha)$ . It is worth noting that definition (2.1) does not depend on  $\beta$ , see (1.2). A family of operators  $(L^\Delta(t))_{t \geq 0} \subset L(\mathbb{B})$  is said to be strongly continuous in  $t$  if  $t \longmapsto L^\Delta(t)k \in \mathbb{B}_\alpha$  is continuous for any  $k \in \mathbb{B}_{\alpha'}$  and  $\alpha' < \alpha$ . This family is said to be continuous in  $t$  w.r.t. the uniform topology if  $t \longmapsto L^\Delta(t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$  is continuous for any  $\alpha' < \alpha$  in the uniform operator topology. The next lemma collects some basic properties. Its proof can be obtained by elementary arguments and is therefore omitted.

**Lemma 2.1.** *Let  $W^\Delta(t, s)$  be forward regularity preserving evolution system on  $\mathbb{B}$  and  $(L^\Delta(t))_{t \geq 0}$  its generator, see (1.3) and (1.4). Then the following assertions hold:*

1.  $W^\Delta(t, s) \in L(\mathbb{B})$  is uniquely determined by its generator.
2.  $(t, s) \longmapsto W^\Delta(t, s) \in L(\mathbb{B})$  is continuous w.r.t. the uniform topology on  $L(\mathbb{B})$ .
3. Suppose that the generator is continuous in  $t$  w.r.t. the uniform topology. Then  $W^\Delta(t, s)$  is continuously differentiable in  $(t, s)$  w.r.t. the uniform topology.

4. Let  $\widetilde{W}^\Delta(t, s)$  be another forward evolution system with generator  $\widetilde{L}^\Delta(t)$  and suppose that  $L^\Delta(t)$  and  $\widetilde{L}^\Delta(t)$  are continuous in  $t$  w.r.t. the uniform topology. Then for any  $\alpha' < \alpha_0 < \alpha_1 < \alpha$  and  $T > 0$  there exist a constant  $A(\alpha', \alpha_0, \alpha_1, \alpha, T) > 0$  such that

$$\|W^\Delta(t, s) - \widetilde{W}^\Delta(t, s)\|_{\alpha'\alpha} \leq A \int_s^t \|L^\Delta(r) - \widetilde{L}^\Delta(r)\|_{\alpha_0\alpha_1} dr$$

is satisfied.

A sufficient condition for the existence of forward and backward evolution systems on the scale  $\mathbb{B}$  is given in the statement below.

**Theorem 2.2.** Let  $(L^\Delta(t))_{t \geq 0} \subset L(\mathbb{B})$  be continuous in  $t$  w.r.t. the uniform topology and suppose that there exists a family of operators  $(S_t(r))_{r \geq 0} \subset L(\mathbb{B})$ ,  $t \geq 0$  such that  $S_t(r_1)S_t(r_2) = S_t(r_1 + r_2)$  holds for all  $t, r_1, r_2 \geq 0$ . Moreover, suppose that the following conditions hold:

- (a) For any  $k \in \mathbb{B}_{\alpha'}$ ,  $\alpha' < \alpha$  and  $t \geq 0$ ,  $r \mapsto S_t(r)k \in \mathbb{B}_\alpha$  is continuously differentiable and satisfies

$$\frac{\partial}{\partial r} S_t(r)k = L^\Delta(t)S_t(r)k = S_t(r)L^\Delta(t)k, \quad r \geq 0.$$

- (b) There exist constants  $M(\alpha', \alpha) \geq 1$  and  $\omega(\alpha', \alpha) \in \mathbb{R}$  such that

$$\|S_{t_n}(s_n) \cdots S_{t_1}(s_1)\|_{\alpha'\alpha} \leq M(\alpha', \alpha) e^{\omega(\alpha', \alpha) \sum_{j=1}^n s_j}$$

holds, where  $n \in \mathbb{N}$ ,  $s_1, \dots, s_n \geq 0$  and  $0 \leq t_1 \leq \dots \leq t_n$  are arbitrary.

Then there exists a forward regularity preserving evolution system  $W^\Delta(t, s)$  in the scale  $\mathbb{B}$  such that  $L^\Delta(t)$  is its generator and for all  $\alpha' < \alpha$  and  $0 \leq s \leq t$

$$\|W^\Delta(t, s)\|_{\alpha'\alpha} \leq M(\alpha', \alpha) e^{\omega(\alpha', \alpha)(t-s)} \quad (2.2)$$

is satisfied.

*Proof.* Fix  $T > 0$  and define for  $n \in \mathbb{N}$  piecewise constant operators  $A_n(t)$  by setting  $t_k^n = \frac{k}{n}T$  and

$$\begin{cases} A_n(t) = A(t_k^n), & t_k^n \leq t < t_{k+1}^n, \quad k = 0, \dots, n-1 \\ A_n(T) = A(T) \end{cases}.$$

Moreover, let  $W_n^\Delta(t, s)$  be given by

$$W_n^\Delta(t, s) := \begin{cases} S_{t_j^n}(t - s), & t_j^n \leq s \leq t \leq t_{j+1}^n \\ S_{t_k^n}(t - t_k^n) S_n(l, k) S_{t_l^n}(t_{l+1}^n - s), & k > l, \ t_k^n \leq t \leq t_{k+1}^n, \ t_l^n \leq s \leq t_{l+1}^n \end{cases}, \quad (2.3)$$

where  $S_n(l, k) := S_{t_{k-1}^n}(\frac{T}{n}) \cdots S_{t_{l+1}^n}(\frac{T}{n})$  is time ordered in such a way that smaller times stand to the right. Similar arguments to [Paz83, Chapter 5, Theorem 3.1] show that there exists a family of operators  $W^\Delta(t, s) \in L(\mathbb{B})$  such that

$$\lim_{n \rightarrow \infty} W_n^\Delta(t, s)k = W^\Delta(t, s)k, \quad k \in \mathbb{B}_{\alpha'} \quad (2.4)$$

holds in  $\mathbb{B}_\alpha$  uniformly on compacts in  $(t, s)$ . This family of operators satisfies, by construction, (2.2). Properties (1.3), (1.4) and (1.5) follow by similar arguments to [Paz83, Chapter 5, Theorem 3.1, Theorem 4.3].  $\square$

**Remark 2.3.** *A similar statement can be shown for backward evolution systems  $Q^\Delta(s, t)$ , provided condition (a) and*

(b') *There exist constants  $M(\alpha', \alpha) \geq 1$  and  $\omega(\alpha', \alpha) \in \mathbb{R}$  with*

$$\|S_{t_1}(s_1) \cdots S_{t_n}(s_n)\|_{\alpha'\alpha} \leq M(\alpha', \alpha) e^{\omega(\alpha', \alpha) \sum_{j=1}^n s_j}$$

*holds, where  $n \in \mathbb{N}$ ,  $s_1, \dots, s_n \geq 0$  and  $0 \leq t_1 \leq \dots \leq t_n$  are arbitrary.*

*are satisfied.*

A classical solution to (1.6) is a function  $u \in C^1([s, \infty); \mathbb{B}_\alpha)$  such that (1.6) holds. Similarly, a classical solution to (1.7) is a function  $v \in C^1([0, T^*]; \mathbb{B}_\alpha)$  such that (1.7) holds. Uniqueness of classical solutions is stated below and follows by a modification of [Paz83, Chapter 5, Theorem 4.2].

**Theorem 2.4.** *Suppose that the same conditions as in Theorem 2.2 are fulfilled and let  $W^\Delta(t, s)$ ,  $Q^\Delta(s, t)$  be the forward and backward regularity preserving evolution systems associated to  $L^\Delta(t)$ . Then for every  $k \in \mathbb{B}_{\alpha'}$  equation (1.6) has a unique classical solution given by  $u(t) = W^\Delta(t, s)x$ , and equation (1.7) has a unique classical solution given by  $v(s) = Q^\Delta(s, t)x$ .*

The next statement relates the constructed forward and backward evolution systems to solutions of the dual Cauchy problems given below. Let  $\mathbb{B}^* = (\mathbb{B}_\alpha^*)_{\alpha > \alpha^*}$  be the dual scale of Banach spaces, where  $\mathbb{B}_\alpha^*$  is the dual space to  $\mathbb{B}_\alpha$ . Then  $\mathbb{B}_\alpha^* \ni \ell \mapsto \ell|_{\mathbb{B}_{\alpha'}} \in \mathbb{B}_{\alpha'}^*$  defines an embedding such that  $\|\ell|_{\mathbb{B}_{\alpha'}}\|_{\mathbb{B}_{\alpha'}^*} \leq \|\ell\|_{\mathbb{B}_\alpha^*}$  holds for all  $\alpha' < \alpha$ . For  $k \in \mathbb{B}_\alpha$  and

$\ell \in \mathbb{B}_\alpha^*$  let  $\langle k, \ell \rangle = \ell(k)$  be the dual pairing and denote by  $W^\Delta(s, t)^*$  and  $Q^\Delta(t, s)^*$  the adjoint operators defined on the scale  $\mathbb{B}^*$ . These operators satisfy

$$Q^\Delta(t, r)^* Q^\Delta(r, s)^* = Q^\Delta(t, s)^*, \quad W^\Delta(s, r)^* W^\Delta(r, t)^* = W^\Delta(s, t)^*,$$

and hence  $Q^\Delta(t, s)^*$  is a forward evolution system whereas  $W^\Delta(s, t)^*$  is a backward evolution system on the scale  $\mathbb{B}^*$ . Using (1.6) and (1.7), it follows that they satisfy for any  $\alpha' < \alpha$  and  $\ell \in \mathbb{B}_\alpha^*$  the equations

$$\begin{aligned} \frac{d}{ds} \langle k, W^\Delta(s, t)^* \ell \rangle &= -\langle L^\Delta(s)k, W^\Delta(s, t)^* \ell \rangle, \quad k \in \mathbb{B}_{\alpha'}, \quad s \in [0, t) \\ \frac{d}{dt} \langle k, Q^\Delta(t, s)^* \ell \rangle &= \langle L^\Delta(t)k, Q^\Delta(t, s)^* \ell \rangle, \quad k \in \mathbb{B}_{\alpha'}, \quad t \in [s, \infty). \end{aligned}$$

**Remark 2.5.** Suppose that  $L^\Delta(t) \in L(\mathbb{B})$  is continuous in  $t$  w.r.t. the uniform topology. Then  $W^\Delta(s, t)^*$  and  $Q^\Delta(t, s)^*$  are both continuously differentiable in  $(s, t)$  w.r.t. the uniform topology.

Denote by  $\sigma(\mathbb{B}_{\alpha'}^*, \mathbb{B}_{\alpha'})$  the topology generated by the seminorms  $p_k(\ell) := |\langle k, \ell \rangle|$ ,  $\ell \in \mathbb{B}_{\alpha'}^*$  and  $k \in \mathbb{B}_{\alpha'}$ . The next statement follows by a modification of the arguments given in [Kol13].

**Theorem 2.6.** Suppose that the same conditions as for Theorem 2.2 are satisfied. Let  $\alpha' < \alpha$  and  $\ell \in \mathbb{B}_\alpha^*$  be arbitrary. Then the following holds:

1. Let  $t > 0$  and  $(\ell(s))_{s \in [0, t]} \subset \mathbb{B}_{\alpha'}^*$  be continuous in  $s$  w.r.t.  $\sigma(\mathbb{B}_{\alpha'}^*, \mathbb{B}_{\alpha'})$  such that

$$\frac{d}{ds} \langle k, \ell(s) \rangle = -\langle L^\Delta(s)k, \ell(s) \rangle, \quad \ell(t) = \ell, \quad k \in \mathbb{B}_{\alpha'}, \quad s \in [0, t) \quad (2.5)$$

is satisfied. Then  $\ell(s) = W^\Delta(s, t)^* \ell$  holds for any  $s \in [0, t]$ .

2. Let  $s \geq 0$  and  $(\ell(t))_{t \in [s, \infty)} \subset \mathbb{B}_{\alpha'}^*$  be continuous in  $t$  w.r.t.  $\sigma(\mathbb{B}_{\alpha'}^*, \mathbb{B}_{\alpha'})$  such that

$$\frac{d}{dt} \langle k, \ell(t) \rangle = \langle L^\Delta(t)k, \ell(t) \rangle, \quad \ell(s) = \ell, \quad k \in \mathbb{B}_{\alpha'}, \quad t > s \quad (2.6)$$

is satisfied. Then  $\ell(t) = Q^\Delta(t, s)^* \ell$  holds for any  $t \in [s, \infty)$ .

### 3 Worsening evolution systems

The aim of this section is to prove existence, uniqueness and stability of solutions to (1.6) and (1.7) in the scale  $\mathbb{B}$ . For this purpose we assume that  $L^\Delta(t) = A^\Delta(t) + B^\Delta(t)$ , where  $A^\Delta(t) \in L(\mathbb{B})$  is continuous in  $t$  w.r.t. the uniform topology and  $(B^\Delta(t))_{t \geq 0} \subset L(\mathbb{B})$  is

strongly continuous in  $t$  such that there exists an increasing continuous function  $M(\alpha)$  satisfying for all  $\alpha' < \alpha$

$$\|B^\Delta(t)\|_{\alpha'\alpha} \leq \frac{M(\alpha)}{\alpha - \alpha'}, \quad t \geq 0. \quad (3.1)$$

The operator  $B^\Delta(t)$  is said to be an Ovcyannikov operator.

## Forward evolution system

For this section we suppose that there exists a (forward) regularity preserving evolution system  $V^\Delta(t, s)$  on  $\mathbb{B}$  which is associated to  $A^\Delta(t)$ . In particular,

$$\frac{\partial}{\partial t} V^\Delta(t, s)k = A^\Delta(t)V^\Delta(t, s)k, \quad (3.2)$$

$$\frac{\partial}{\partial s} V^\Delta(t, s)k = -V^\Delta(t, s)A^\Delta(s)k \quad (3.3)$$

hold in  $\mathbb{B}_\alpha$  for all  $0 \leq s \leq t$  and  $k \in \mathbb{B}_{\alpha'}$ , where  $\alpha' < \alpha$ . Suppose that there exist constants  $A \geq 1$  and  $\omega \in \mathbb{R}$  such that for all  $\alpha' < \alpha$

$$\|V^\Delta(t, s)\|_{\alpha'\alpha} \leq Ae^{\omega(t-s)}, \quad 0 \leq s \leq t \quad (3.4)$$

holds. Define  $T(\alpha', \alpha) := \frac{\alpha - \alpha'}{2AeM(\alpha)}$ .

**Theorem 3.1.** *There exists a unique (forward) worsening evolution system associated to  $L^\Delta(t) = A^\Delta(t) + B^\Delta(t)$ . Namely, there exists a family of operators  $W^\Delta(t, s) : \mathbb{B} \longrightarrow \mathbb{B}$ ,  $0 \leq s \leq t$  such that the following holds:*

1.  $W^\Delta(t, s) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$  for any  $\alpha' < \alpha$  and  $0 \leq t - s < T(\alpha', \alpha)$  such that

$$\|W^\Delta(t, s)\|_{\alpha'\alpha} \leq e^{\omega(t-s)} \frac{T(\alpha', \alpha)}{T(\alpha', \alpha) - (t - s)}$$

*is satisfied and property (1.5) holds.*

2. *For any  $\alpha' < \alpha$  and  $k \in \mathbb{B}_{\alpha'}$ ,  $(s, t) \longmapsto W^\Delta(t, s)k$  is continuously differentiable in  $\mathbb{B}_\alpha$  such that for all  $0 \leq t - s < T(\alpha', \alpha)$*

$$\frac{\partial}{\partial t} W^\Delta(t, s)k = (A^\Delta(t) + B^\Delta(t))W^\Delta(t, s)k, \quad (3.5)$$

$$\frac{\partial}{\partial s} W^\Delta(t, s)k = -W^\Delta(t, s)(A^\Delta(s) + B^\Delta(s))k \quad (3.6)$$

*hold in  $\mathbb{B}_\alpha$ .*



3. Fix  $s \geq 0$ ,  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$  and suppose that there exists  $T > 0$  and a function  $u \in C([s, s+T]; \mathbb{B}_{\alpha}) \cap C^1((s, s+T); \mathbb{B}_{\alpha})$  such that for all  $s \leq t < s+T$  (1.6) is satisfied. Then  $u(t) = W^{\Delta}(t, s)k$  holds for any  $s \leq t < s + \min\{T, T(\alpha', \alpha)\}$ .

*Proof.* Define a sequence of operators  $(W_n^{\Delta}(t, s))_{0 \leq s \leq t} \subset L(\mathbb{B})$  by  $W_0^{\Delta}(t, s)k = V^{\Delta}(t, s)k$  and

$$W_{n+1}^{\Delta}(t, s)k := \int_s^t V^{\Delta}(t, r)B^{\Delta}(r)W_n^{\Delta}(r, s)k dr \quad (3.7)$$

for  $k \in \mathbb{B}_{\alpha'}$ . Then for any  $\alpha' < \alpha$ ,  $n \geq 0$  and  $k \in \mathbb{B}_{\alpha'}$  the function  $W_n^{\Delta}(t, s)k$  is continuous in  $\mathbb{B}_{\alpha}$  and satisfies

$$\|W_n^{\Delta}(t, s)k\|_{\alpha} \leq \|k\|_{\alpha'} e^{\omega(t-s)} \left( \frac{t-s}{T(\alpha', \alpha)} \right)^n.$$

In fact, let  $\alpha_j := \alpha' + j \frac{\alpha - \alpha'}{2n}$ ,  $j = 0, \dots, 2n$  and for  $s \leq t_1 \leq \dots \leq t_n \leq t$

$$Q_n(t, t_1, \dots, t_n, s)k := V^{\Delta}(t, t_1)B^{\Delta}(t_1) \cdots V^{\Delta}(t_{2n-2}, t_{2n-1})B^{\Delta}(t_{2n-1})V^{\Delta}(t_{2n}, s)k.$$

Then by (3.4) and (3.1) we obtain

$$\begin{aligned} \|W_n^{\Delta}(t, s)k\|_{\alpha} &\leq \int_s^t \cdots \int_s^{t_{n-1}} \|Q_n(t, t_1, \dots, t_n, s)k\|_{\alpha} dt_n \dots dt_1 \\ &\leq A^n e^{\omega(t-s)} \|k\|_{\alpha'} \frac{(2n)^n}{(\alpha - \alpha')^n} \int_s^t \cdots \int_s^{t_{n-1}} \prod_{j=0}^{n-1} M(\alpha_{2j+1}) dt_n \dots dt_1 \\ &\leq \|k\|_{\alpha'} e^{\omega(t-s)} \frac{(t-s)^n}{n!} \frac{(2M(\alpha)nA)^n}{(\alpha - \alpha')^n} \leq \|k\|_{\alpha'} e^{\omega(t-s)} \left( \frac{2eAM(\alpha)(t-s)}{\alpha - \alpha'} \right)^n, \end{aligned}$$

where we have used the Stirling formula. Choose  $q \in (0, 1)$ , then we obtain for any  $0 \leq t-s \leq qT(\alpha', \alpha)$

$$\|W_n^{\Delta}(t, s)k\|_{\alpha} \leq \|k\|_{\alpha'} e^{\omega(t-s)} q^n$$

and hence the series  $\sum_{n=0}^{\infty} W_n^{\Delta}(t, s)k =: W^{\Delta}(t, s)k$  converges uniform. Since  $q$  was arbitrary, it follows that  $W^{\Delta}(t, s)k$  is continuous in  $(t, s)$  for  $t-s < T(\alpha', \alpha)$  and satisfies

$$\begin{aligned} \|W^{\Delta}(t, s)k\|_{\alpha} &\leq \sum_{n=0}^{\infty} \|W_n^{\Delta}(t, s)k\|_{\alpha} \leq \|k\|_{\alpha'} e^{\omega(t-s)} \sum_{n=0}^{\infty} \left( \frac{t-s}{T(\alpha', \alpha)} \right)^n \\ &= \|k\|_{\alpha'} e^{\omega(t-s)} \frac{T(\alpha', \alpha)}{T(\alpha', \alpha) - (t-s)}. \end{aligned}$$

We show that  $W^\Delta(t, s)$  is differentiable. Take  $\alpha_j := \alpha' + j \frac{\alpha - \alpha'}{2(n+1)}$ ,  $j = 0, \dots, 2(n+1)$ , then we obtain for  $s \leq r \leq t$  that

$$\begin{aligned} \|V^\Delta(t, r)B^\Delta(r)W_n^\Delta(r, s)k\|_\alpha &\leq e^{\omega(t-s)}(AM(\alpha))^{n+1} \frac{2(n+1)}{\alpha - \alpha'} \frac{(t-s)^n}{n!} \frac{(2(n+1))^n}{(\alpha - \alpha')^n} \|k\|_{\alpha'} \\ &\leq e^{\omega(t-s)} \|k\|_{\alpha'} \frac{4eAM(\alpha)}{\alpha - \alpha'} n \left( \frac{t-s}{T(\alpha', \alpha)} \right)^n \end{aligned}$$

is satisfied. So for any  $s \leq r \leq t$  and  $q \in (0, 1)$  such that  $|t-s| \leq qT(\alpha', \alpha)$  the series  $\sum_{n=0}^{\infty} V^\Delta(t, r)B^\Delta(r)W_n^\Delta(r, s)k$  is uniformly convergent. For  $t-s < T(\alpha', \alpha)$  we find  $\alpha'' \in (\alpha', \alpha)$  such that  $t-s < T(\alpha', \alpha'')$ , hence  $W^\Delta(r, s)k \in \mathbb{B}_{\alpha''}$  is continuous. Since  $V^\Delta(t, s)B^\Delta(r) \in L(\mathbb{B}_{\alpha''}, \mathbb{B}_\alpha)$  is strongly continuous it follows that

$$\begin{aligned} W^\Delta(t, s)k &= V^\Delta(t, s)k + \sum_{n=1}^{\infty} \int_s^t V^\Delta(t, r)B^\Delta(r)W_{n-1}^\Delta(r, s)k dr \\ &= V^\Delta(t, s)k + \int_s^t V^\Delta(t, r)B^\Delta(r)W^\Delta(r, s)k dr \end{aligned}$$

is fulfilled. Hence  $W^\Delta(t, s)k$  is differentiable w.r.t.  $t$  in  $\mathbb{B}_\alpha$  and differentiating above equality, see (3.2), yields (3.5). The sequence  $(W_n^\Delta(t, s)k)_{n \in \mathbb{N}}$  also satisfies the relation

$$W_{n+1}^\Delta(t, s)k = \int_s^t W_n^\Delta(t, r)B^\Delta(r)V^\Delta(r, s)k dr.$$

A repetition of above arguments shows that  $(W^\Delta(t, s)k)_{0 \leq s \leq t}$  satisfies (3.6). For the last assertion let  $w(t) := W^\Delta(t, s)k - u(t)$ , where  $s \leq t < s + \min\{T, T(\alpha', \alpha)\}$ . Then  $w(s) = 0$  and it suffices to show that  $w = 0$ . Since  $w$  solves (1.6) it follows that  $w$  satisfies for any  $s \leq t < s + \min\{T, T(\alpha', \alpha)\}$  and  $\alpha'' > \alpha$

$$w(t) = \int_s^t V^\Delta(t, r)B^\Delta(r)w(r)dr$$

in  $\mathbb{B}_{\alpha''}$ . Define  $\alpha_j := \alpha + j \frac{\alpha'' - \alpha}{2n}$ ,  $j = 0, \dots, 2n$  and  $C_\alpha := \sup_{r \in [s, t]} \|W^\Delta(r, s)k - u(r)\|_\alpha < \infty$ .

It follows for  $s \leq t_n \leq \dots \leq t_1 \leq t$  and

$$Q(t, t_1, \dots, t_n) := V^\Delta(t, t_1)B^\Delta(t_1) \cdots V^\Delta(t_{n-1}, t_n)B^\Delta(t_n) \quad (3.8)$$

that

$$\|Q(t, t_1, \dots, t_n)w(t_n)\|_{\alpha''} \leq A^n e^{\omega(t-t_n)} \frac{M(\alpha'')^n (2n)^n}{(\alpha'' - \alpha)^n} \|w(t_n)\|_{\alpha} \quad (3.9)$$

holds. Hence we obtain by  $\|w(t_n)\|_{\alpha} \leq C_{\alpha}$  the estimate

$$\begin{aligned} \|w(t)\|_{\alpha''} &\leq \int_s^t \cdots \int_s^{t_{n-1}} \|Q(t, t_1, \dots, t_n, s)w(t_n)\|_{\alpha''} dt_n \cdots dt_1 \\ &\leq C_{\alpha} e^{\omega(t-s)} \left( \frac{2eAM(\alpha'')(t-s)}{\alpha'' - \alpha} \right)^n, \end{aligned}$$

where we have assumed w.l.g. that  $\omega \geq 0$ . This implies  $w(t) = 0$  in  $\mathbb{B}_{\alpha''}$  and hence  $\mathbb{B}_{\alpha}$  for

$$s \leq t < s + \min \left\{ T, T(\alpha', \alpha), \frac{\alpha'' - \alpha}{2eAM(\alpha'')} \right\}.$$

Applying above arguments to  $\alpha'' = \alpha + 1$  shows for any  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$  that (1.6) is unique on  $[s, s + T_0(\alpha', \alpha)q]$  for any  $q \in (0, 1)$  and  $T_0(\alpha', \alpha) := \min\{T(\alpha', \alpha), \frac{1}{2eAM(\alpha+1)}\}$ . Changing  $s$  to  $s + T_0(\alpha', \alpha)q$  and iterating this procedure yields the assertion. Such an iteration is possible since  $w(s + qT_0(\alpha', \alpha)) = 0 \in \mathbb{B}_{\alpha'}$ . Property (1.5) follows by uniqueness.  $\square$

**Remark 3.2.** Previous proof shows that if  $B^{\Delta}(t)$  is continuous in  $t$  w.r.t. the uniform topology, then  $W^{\Delta}(t, s)$  is continuously differentiable in  $(t, s)$  w.r.t. the uniform topology.

For any  $n \in \mathbb{N}$ , let  $(A_n^{\Delta}(t))_{t \geq 0}$  be continuous in  $t$  w.r.t. the uniform topology in the scale  $\mathbb{B}$  and  $(V_n^{\Delta}(t, s))_{0 \leq s \leq t}$  the associated regularity preserving (forward) evolution systems. Suppose that there exists constants  $A \geq 1$  and  $\omega \in \mathbb{R}$  such that

$$\|V_n^{\Delta}(t, s)\|_{\alpha'\alpha} \leq A e^{\omega(t-s)}, \quad 0 \leq s \leq t, \quad \alpha' < \alpha, \quad n \in \mathbb{N} \quad (3.10)$$

holds. Let  $(B_n^{\Delta}(t))_{t \geq 0} \subset L(\mathbb{B})$  be strongly continuous in  $t$  for any  $n \in \mathbb{N}$  such that there exists an increasing continuous function  $M(\alpha)$  independent of  $n$  and it satisfies

$$\|B_n^{\Delta}(t)\|_{\alpha'\alpha} \leq \frac{M(\alpha)}{\alpha - \alpha'}, \quad \alpha' < \alpha, \quad t \geq 0, \quad n \in \mathbb{N}. \quad (3.11)$$

**Theorem 3.3.** Suppose that there exist operators  $A^{\Delta}(t), V^{\Delta}(t, s), B^{\Delta}(t)$  which satisfy the conditions of Theorem 3.1 with  $M(\alpha)$  as in (3.11). Assume that for any  $T > 0$  and  $\alpha' < \alpha$

$$\sup_{t \in [0, T]} \|B_n^{\Delta}(t) - B^{\Delta}(t)\|_{\alpha'\alpha} \longrightarrow 0, \quad n \rightarrow \infty \quad (3.12)$$

and

$$\sup_{t \in [0, T]} \|A_n^\Delta(t) - A^\Delta(t)\|_{\alpha' \alpha} \longrightarrow 0, \quad n \rightarrow \infty \quad (3.13)$$

are satisfied. Then for any  $n \in \mathbb{N}$  there exist  $W^{\Delta, n}(t, s)$  and  $W^\Delta(t, s)$  corresponding to  $(A_n^\Delta(t), B_n^\Delta(t))$  and  $(A^\Delta(t), B^\Delta(t))$  respectively, with the properties stated in Theorem 3.1. Moreover, for any  $\alpha' < \alpha$  and  $k \in \mathbb{B}_{\alpha'}$

$$W^{\Delta, n}(t, s)k \longrightarrow W^\Delta(t, s)k, \quad n \rightarrow \infty$$

holds in  $\mathbb{B}_\alpha$  uniformly on compact subsets of  $\{(t, s) \mid 0 \leq t - s < T(\alpha', \alpha)\}$ .

*Proof.* Lemma 2.1 together with (3.13) shows that

$$\|V_n^\Delta(t, s) - V^\Delta(t, s)\|_{\alpha' \alpha} \longrightarrow 0, \quad n \rightarrow \infty \quad (3.14)$$

holds uniformly on compacts for  $0 \leq s \leq t$ . Therefore without loss of generality, we can assume that  $V^\Delta(t, s)$  satisfies (3.10) with the same constants. Estimates (3.10) and (3.11), Theorem 3.1 and (3.7) imply that  $W^{\Delta, n}(t, s), W^\Delta(t, s)$  exist and are given by  $W^\Delta(t, s) = \sum_{k=0}^{\infty} W_k^\Delta(t, s)$  and  $W^{\Delta, n}(t, s) = \sum_{k=0}^{\infty} W_k^{\Delta, n}(t, s)$ , respectively. Moreover, from (3.10) and (3.11) it follows that

$$\|W_k^{\Delta, n}(t, s)\|_{\alpha' \alpha} \leq e^{\omega(t-s)} \left( \frac{t-s}{T(\alpha', \alpha)} \right)^k$$

and hence the series converges uniformly w.r.t.  $(t, s)$  and  $n$ , whenever  $(t, s)$  are restricted to a compact with  $0 \leq t - s < T(\alpha', \alpha)$ . Thus it suffices to show  $W_k^{\Delta, n}(t, s) \longrightarrow W_k^\Delta(t, s)$ ,  $n \rightarrow \infty$  in  $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$  for any  $k \in \mathbb{N}$ . For  $k = 0$  this follows from (3.14) and for  $k \geq 1$  by induction and (3.12).  $\square$

**Remark 3.4.** Clearly it is not necessary to assume that (3.12) and (3.13) hold for each  $T > 0$ . It is enough to check the convergence on  $[s, s + qT(\alpha', \alpha)]$  for all  $s \geq 0$  and  $q \in (0, 1)$ .

Next we show continuous dependence on  $L^\Delta$  without assuming existence of a worsening evolution system for the limiting equation.

**Theorem 3.5.** Let  $(L^\Delta(t))_{t \geq 0}, (L_0^\Delta(t))_{t \geq 0} \subset L(\mathbb{B})$  be continuous in  $t$  w.r.t. the uniform topology. Suppose that the same conditions as in Theorem 3.1 are satisfied and let  $(W^\Delta(t, s))_{0 \leq t-s < T(\alpha', \alpha)}$  be the local worsening evolution system associated with  $L^\Delta(t)$ . Fix  $\alpha' > \alpha_*$ ,  $s \geq 0$ ,  $T > 0$  and  $r_0 \in \mathbb{B}_{\alpha'}$ . Suppose that there exists  $(r_{t,s})_{s \leq t \leq s+T} \subset \mathbb{B}_{\alpha'}$  being continuous in  $(t, s)$  w.r.t. the norm in  $\mathbb{B}_{\alpha'}$  and for any  $\alpha > \alpha'$ ,  $t \mapsto r_{t,s} \in \mathbb{B}_\alpha$  is continuously differentiable with

$$\frac{\partial r_{t,s}}{\partial t} = L_0^\Delta(t)r_{t,s}, \quad r_{s,s} = r_0.$$

Then for every  $\alpha > \alpha'$  and compact  $K \subset \{(t, s) \mid 0 \leq t - s < T(\alpha', \alpha)\}$  there exists  $\alpha'' \in (\alpha', \alpha)$  and another compact  $I \subset [0, \infty)$  such that for all  $(t, s) \in K$

$$\|W^\Delta(t, s)r_0 - r_{t,s}\|_\alpha \leq \sup_{\tau \in I} \|L_0^\Delta(\tau) - L^\Delta(\tau)\|_{\alpha'\alpha''} \sup_{\tau \in I} \|r_{\tau,s}\|_{\alpha'} h(t-s)$$

holds for an analytic function  $h(t-s)$  having a pole of second order at  $T(\alpha', \alpha)$  and  $h(0) = 0$ .

*Proof.* Observe that  $\alpha' \mapsto T(\alpha', \alpha) = \frac{\alpha - \alpha'}{2eAM(\alpha)}$  is continuous and decreasing. Hence we can find  $\alpha'' \in (\alpha', \alpha)$  such that for all  $(t, s) \in K$  we have  $0 \leq t - s < T(\alpha'', \alpha) < T(\alpha', \alpha)$ . The function  $[s, t] \ni \tau \mapsto W^\Delta(t, \tau)r_{\tau,s} \in \mathbb{B}_\alpha$  is continuously differentiable such that

$$\frac{\partial}{\partial \tau} W^\Delta(t, \tau)r_{\tau,s} = W^\Delta(t, \tau)(L_0^\Delta(\tau) - L^\Delta(\tau))r_{\tau,s}.$$

Here  $L_0^\Delta(\tau), L^\Delta(\tau) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_{\alpha''})$  and hence the derivative is continuous in  $\tau$  w.r.t. the norm in  $\mathbb{B}_\alpha$ . Then

$$\begin{aligned} \|W^\Delta(t, s) - r_{t,s}\|_\alpha &\leq \int_s^t \|W^\Delta(t, \tau)(L_0^\Delta(\tau) - L^\Delta(\tau))r_{\tau,s}\|_\alpha d\tau \\ &\leq \sup_{\tau \in [s, t]} \|L_0^\Delta(\tau) - L^\Delta(\tau)\|_{\alpha'\alpha''} \int_s^t \frac{T(\alpha'', \alpha)}{T(\alpha'', \alpha) - (t - \tau)} \|r_{\tau,s}\|_{\alpha'} d\tau \\ &\leq \sup_{\tau \in [s, t]} \|L_0^\Delta(\tau) - L^\Delta(\tau)\|_{\alpha'\alpha''} \sup_{\tau \in [s, t]} \|r_{\tau,s}\|_{\alpha'} h(t-s), \end{aligned}$$

where  $h(t-s) := \frac{T(\alpha'', \alpha)}{(T(\alpha'', \alpha) - (t-s))^2} - \frac{1}{T(\alpha'', \alpha)}$  implies the assertion.  $\square$

**Remark 3.6.** Note that only existence of  $r_{t,s}$  is assumed and we do not require to have uniqueness. Moreover, existence is assumed only for one fixed choice of  $r_0$ ,  $s$  and  $\alpha'$ .

## Backward evolution system

For this section we suppose that there exists a regularity preserving backward evolution system  $(U^\Delta(s, t))_{0 \leq s \leq t} \subset L(\mathbb{B})$  having generator  $A^\Delta(t)$ . Moreover, assume that there exist  $A \geq 1$  and  $\omega \in \mathbb{R}$  such that for all  $\alpha' < \alpha$

$$\|U^\Delta(s, t)\|_{\alpha'\alpha} \leq Ae^{\omega(t-s)}, \quad 0 \leq s \leq t$$

holds. The next statement is proved analogously to Theorem 3.1.

**Theorem 3.7.** *There exists a unique worsening backward evolution system  $(Q^\Delta(s, t))_{0 \leq s \leq t}$  such that the following holds:*

1.  $Q^\Delta(s, t) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$  for any  $\alpha' < \alpha$  and  $0 \leq t - s < T(\alpha', \alpha)$  such that

$$\|Q^\Delta(s, t)\|_{\alpha'\alpha} \leq e^{\omega(t-s)} \frac{T(\alpha', \alpha)}{T(\alpha', \alpha) - (t - s)}$$

is satisfied.

2. For  $\alpha' < \alpha$  and  $s \leq t$  let  $\alpha(s, t, \alpha') := \inf \{\beta \geq \alpha' \mid Q^\Delta(s, t)k \in \mathbb{B}_\beta\}$ . Then for any  $0 \leq t - s < T(\alpha', \alpha)$  and  $0 \leq r - s < T(\alpha(r, t, \alpha'), \alpha)$

$$Q^\Delta(s, t)k = Q^\Delta(s, r)Q^\Delta(r, t)k \quad (3.15)$$

holds for all  $k \in \mathbb{B}_{\alpha'}$ .

3. For any  $\alpha' < \alpha$  and  $k \in \mathbb{B}_{\alpha'}$ ,  $Q^\Delta(s, t)k$  is continuously differentiable in  $\mathbb{B}_\alpha$  such that for all  $0 \leq t - s < T(\alpha', \alpha)$

$$\frac{\partial}{\partial s} Q^\Delta(s, t)k = -(A^\Delta(s) + B^\Delta(s))Q^\Delta(s, t)k, \quad (3.16)$$

$$\frac{\partial}{\partial t} Q^\Delta(s, t)k = Q^\Delta(s, t)(A^\Delta(t) + B^\Delta(t))k \quad (3.17)$$

hold in  $\mathbb{B}_\alpha$ .

4. Fix  $T^* > 0$ ,  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$  and suppose that there exists  $u \in C((0, T^*]; \mathbb{B}_\alpha) \cap C^1((0, T^*); \mathbb{B}_\alpha)$  such that for all  $s \in (0, T^*)$  (1.7) is satisfied. Then  $u(s) = Q^\Delta(s, T^*)k$  holds for any  $T^* - T(\alpha', \alpha) < s \leq T^*$ .

Continuous dependence on  $A^\Delta(t)$  and  $B^\Delta(t)$  can be shown similarly to the case of an forward worsening evolution system.

## Positivity preserving evolution systems

Suppose that  $\mathbb{B}$  is a scale of ordered Banach spaces. Namely, for each  $\alpha > \alpha_*$  the space  $\mathbb{B}_\alpha$  is an ordered Banach space. The order is assumed to be compatible with the scale  $\mathbb{B}$ , i.e. for all  $\alpha' < \alpha$  and  $k, h \in \mathbb{B}_{\alpha'}$

$$k \leq_{\alpha'} h \Leftrightarrow k \leq_\alpha h,$$

where  $\leq_\alpha$  denotes the order on  $\mathbb{B}_\alpha$  and  $\leq_{\alpha'}$  the order on  $\mathbb{B}_{\alpha'}$ . Thus we can omit the dependence on  $\alpha$ . For details on ordered Banach spaces we refer to [BA06] and the references therein.

Given  $C \in L(\mathbb{B})$ , we say that  $C$  is positive if for each  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$ :  $k \geq 0$  implies  $Ck \geq 0$ . The next theorem establishes a comparison principle for the constructed solutions.

**Theorem 3.8.** Suppose that  $A^\Delta(t)$  and  $V^\Delta(t, s)$  are given as in Theorem 3.1 and  $V^\Delta(t, s)$  is positive. Let  $(B_0^\Delta(t))_{t \geq 0}, (B_1^\Delta(t))_{t \geq 0} \subset L(\mathbb{B})$  be two positive operators. Assume that  $t \mapsto B_j^\Delta(t) \in L(\mathbb{B})$  are strongly continuous in the scale  $\mathbb{B}$  for  $j = 0, 1$  and there exist continuous increasing functions  $M_0(\alpha), M_1(\alpha) > 0$  satisfying for all  $\alpha' < \alpha$  and  $t \geq 0$

$$\|B_j^\Delta(t)\|_{\alpha'\alpha} \leq \frac{M_j(\alpha)}{\alpha - \alpha'}, \quad j = 0, 1.$$

Let  $(W_0^\Delta(t, s))_{0 \leq s \leq t}$  be the forward worsening evolution system associated with  $A^\Delta(t) + B_0^\Delta(t)$  and  $(W_1^\Delta(t, s))_{0 \leq s \leq t}$  be the forward worsening evolution system associated with  $A^\Delta(t) + B_0^\Delta(t) - B_1^\Delta(t)$ . Suppose that  $W_1^\Delta(t, s)$  is positive. Then for any  $\alpha' < \alpha < \alpha''$  and  $0 \leq k \in \mathbb{B}_{\alpha'}$

$$W_1^\Delta(t, s)k \leq W_0^\Delta(t, s)k \tag{3.18}$$

holds for all  $s \leq t < s + \min \left\{ \frac{\alpha - \alpha'}{2eA(M_0(\alpha) + M_1(\alpha))}, \frac{\alpha'' - \alpha}{2eAM_0(\alpha'')} \right\}$ .

*Proof.* Let  $w(t) := W_0^\Delta(t, s)k - W_1^\Delta(t, s)k$ . The proof of Theorem 3.1 implies that

$$\begin{aligned} w(t) &= \int_s^t V^\Delta(t, r)B_0^\Delta(r)w(r)dr + \int_s^t V^\Delta(t, r)B_1^\Delta(r)W_1^\Delta(r, s)kdr \\ &\geq \int_s^t V^\Delta(t, r)B_0^\Delta(r)w(r)dr \end{aligned}$$

holds for  $s \leq t < s + \frac{\alpha - \alpha'}{2eA(M_0(\alpha) + M_1(\alpha))}$  in  $\mathbb{B}_{\alpha''}$ , see (3.5). Iterating this inequality yields for any  $n \in \mathbb{N}$  in  $\mathbb{B}_{\alpha''}$

$$W_0^\Delta(t, s)k - W_1^\Delta(t, s)k \geq \int_s^t \cdots \int_s^{t_{n-1}} Q(t, t_1, \dots, t_n, s)w(t_n)dt_n \cdots dt_1 =: I_n,$$

where  $Q(t, t_1, \dots, t_n, s) := V^\Delta(t, t_1)B_0^\Delta(t_1) \cdots V^\Delta(t_{n-1}, t_n)B_0^\Delta(t_n)$ . Let  $\alpha_j := \alpha + j \frac{\alpha'' - \alpha}{2n}$ ,  $j = 0, \dots, 2n$ ,  $C_\alpha := \sup_{r \in [s, t]} \|w(r)\|_\alpha$ , then by (3.9)

$$\|I_n\|_{\alpha''} \leq C_\alpha e^{\omega(t-s)} \left( \frac{2eAM_0(\alpha'')(t-s)}{\alpha'' - \alpha} \right)^n.$$

Hence if  $s \leq t < s + \min \left\{ \frac{\alpha - \alpha'}{2eA(M_0(\alpha) + M_1(\alpha))}, \frac{\alpha'' - \alpha}{2eAM_0(\alpha'')} \right\}$ , then  $I_n \rightarrow 0$ ,  $n \rightarrow \infty$  in  $\mathbb{B}_{\alpha''}$ .  $\square$

## Weak uniqueness

The aim of this section is to prove weak uniqueness for solutions to (1.9) and (1.10). Let  $\mathbb{E}$  be given as in (1.8) and let  $\mathbb{B} = \mathbb{E}^*$  with (1.1). Take  $(A(t))_{t \geq 0} \subset L(\mathbb{E})$  and assume that it is continuous in  $t$  w.r.t. the uniform topology such that conditions (a),(b) and (b') of Theorem 2.2 are satisfied for the scale  $\mathbb{E}$ . Moreover, assume that the constants in conditions (b) and (b') are independent of  $\alpha$ . Then there exists a forward regularity preserving evolution system  $U(t, s)$  and a backward regularity preserving evolution system  $V(s, t)$  on the scale  $\mathbb{E}$ . Both evolution systems are associated with  $A(t)$  and satisfy

$$\|V(t, s)\|_{L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})}, \|U(s, t)\|_{L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})} \leq Ae^{\omega(t-s)}, \quad \alpha' < \alpha, \quad 0 \leq s \leq t \quad (3.19)$$

for some constants  $A \geq 0$  and  $\omega \in \mathbb{R}$ . Let  $B(t) \in L(\mathbb{E})$  be continuous  $t$  w.r.t. the uniform topology and suppose there exists a continuous, increasing function  $M(\alpha) > 0$  such that

$$\|B(t)\|_{L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})} \leq \frac{M(\alpha)}{\alpha - \alpha'}, \quad \alpha' < \alpha, \quad t \geq 0$$

holds. Define  $T(\alpha', \alpha) := \frac{\alpha - \alpha'}{2AeM(\alpha)}$ . The next statement is analogous to Theorem 3.1 and Theorem 3.7.

**Theorem 3.9.** *There exist unique families of operators  $(W(t, s))_{0 \leq s \leq t}$  and  $(Q(s, t))_{0 \leq s \leq t}$  on  $\mathbb{E}$  having the properties:*

1.  $Q(s, t), W(t, s) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$  for any  $\alpha' < \alpha$  and  $0 \leq t - s < T(\alpha', \alpha)$  such that

$$\|Q(s, t)\|_{L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})}, \|W(t, s)\|_{L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})} \leq e^{\omega(t-s)} \frac{T(\alpha', \alpha)}{T(\alpha', \alpha) - (t - s)}$$

is satisfied. Moreover, for any  $x \in \mathbb{E}_\alpha$  and  $\alpha' < \alpha$ ,  $(s, t) \mapsto W(t, s)G \in \mathbb{E}_{\alpha'}$  and  $(s, t) \mapsto Q(s, t)G \in \mathbb{E}_{\alpha'}$  are continuous for  $0 \leq t - s < T(\alpha', \alpha)$ .

2. For any  $\alpha' < \alpha$  and  $G \in \mathbb{E}_\alpha$ ,  $W(t, s)G$  is continuously differentiable in  $\mathbb{E}_{\alpha'}$  such that for all  $0 \leq t - s < T(\alpha', \alpha)$

$$\begin{aligned} \frac{\partial}{\partial t} W(t, s)G &= (A(t) + B(t))W(t, s)G, \\ \frac{\partial}{\partial s} W(t, s)G &= -W(t, s)(A(s) + B(s))G \end{aligned}$$

hold in  $\mathbb{E}_{\alpha'}$ .

3. For any  $\alpha' < \alpha$  and  $x \in \mathbb{E}_\alpha$ ,  $Q(s, t)G$  is continuously differentiable in  $\mathbb{E}_{\alpha'}$  such that for all  $0 \leq t - s < T(\alpha', \alpha)$

$$\begin{aligned} \frac{\partial}{\partial s} Q(s, t)G &= -(A(s) + B(s))Q(s, t)G, \\ \frac{\partial}{\partial t} Q(s, t)G &= Q(s, t)(A(t) + B(t))G \end{aligned}$$

hold in  $\mathbb{E}_{\alpha'}$ .



Let  $A^\Delta(t) := A(t)^*$  and  $B^\Delta(t) := B(t)^*$  be defined on  $\mathbb{B}$ . By duality  $\|B(t)\|_{L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})} = \|B^\Delta(t)\|_{L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)}$  and  $\|A(t)\|_{L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})} = \|A^\Delta(t)\|_{L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)}$  holds for all  $\alpha' < \alpha$ . Moreover, both operators are continuous in  $t$  w.r.t. the uniform topology. By (3.19) it follows that all conditions of Theorem 3.1 and Theorem 3.7 are fulfilled. Denote by  $W^\Delta(t, s)$  and  $Q^\Delta(s, t)$  the associated worsening evolution systems on  $\mathbb{B}$ . Note that they are defined for the same function  $T(\alpha', \alpha)$ .

**Theorem 3.10.** *Let  $\mathcal{Y} \subset \bigcap_{\alpha > \alpha_*} \mathbb{E}_\alpha$  be dense in  $\mathbb{E}_\alpha$  for any  $\alpha$ , then:*

1. *Let  $T > 0$ ,  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$ ,  $s \geq 0$  and let  $(u(t))_{s \leq t < s+T} \subset \mathbb{B}_\alpha$  be continuous in  $t$  w.r.t.  $\sigma(\mathbb{B}_\alpha, \mathbb{E}_\alpha)$  such that*

$$\frac{d}{dt} \langle G, u(t) \rangle = \langle (A(t) + B(t))G, u(t) \rangle, \quad u(s) = k, \quad G \in \mathcal{Y} \quad (3.20)$$

*holds. Then  $u(t) = W^\Delta(t, s)k$  holds for all  $s \leq t < s + \min\{T, T(\alpha', \alpha)\}$ .*

2. *Let  $T > 0$ ,  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$  and  $(v(s))_{s \in (0, T]} \subset \mathbb{B}_\alpha$  be continuous w.r.t.  $\sigma(\mathbb{B}_\alpha, \mathbb{E}_\alpha)$  such that*

$$\frac{d}{ds} \langle G, v(s) \rangle = \langle (A(s) + B(s))G, v(s) \rangle, \quad v(T) = k, \quad G \in \mathcal{Y} \quad (3.21)$$

*holds. Then  $v(s) = Q^\Delta(s, T)k$  holds for all  $0 < s \leq \min\{T, T(\alpha', \alpha)\}$ .*

*Proof.* Let us show the first assertion. Take  $q \in (0, 1)$ ,  $\beta > \beta' > \alpha$  and let  $T_0 := \min\{T, T(\alpha', \alpha), T(\beta', \beta)\}$ . Then for all  $s \leq t \leq s + qT_0$  we have  $G \in \mathbb{E}_\beta$  and hence  $(A(t) + B(t))G \in \mathbb{E}_{\beta'}$ . For  $r \in [s, t]$  let  $g(r) := \langle Q(r, t)G, u(r) \rangle$ , then  $g$  is continuous. Moreover, for  $\delta > 0$  sufficiently small we obtain

$$\begin{aligned} \frac{g(r + \delta) - g(r)}{\delta} &= \left\langle \frac{Q(r + \delta, t)G - Q(r, t)G}{\delta}, u(r) \right\rangle + \left\langle Q(r, t)G, \frac{u(r + \delta) - u(r)}{\delta} \right\rangle \\ &\quad + \left\langle Q(r + \delta, t)G - Q(r, t)G, \frac{u(r + \delta) - u(r)}{\delta} \right\rangle. \end{aligned}$$

Letting  $\delta \rightarrow 0$  yields  $g'(r) = 0$  and hence

$$\langle G, u(t) \rangle = g(t) = g(s) = \langle Q(s, t)G, k \rangle = \langle G, Q(s, t)^*k \rangle,$$

where  $Q(s, t)^* \in L(\mathbb{B}_{\beta'}, \mathbb{B}_\beta)$  is the adjoint operator to  $Q(s, t)$ . Since  $\mathcal{Y}$  is dense it follows that  $Q(s, t)^*k = u(t)$ . Since  $W^\Delta(t, s)$  is also a solution to (3.20) it follows that  $W^\Delta(t, s)k = Q(s, t)^*k = u(t)$ . Moreover,  $(u(t))_{s+qT_0 \leq t < s+T} \subset \mathbb{B}_\alpha$  satisfies

$$\frac{d}{dt} \langle G, u(t) \rangle = \langle (A(t) + B(t))G, u(t) \rangle, \quad u(s + qT_0) = W^\Delta(s + qT_0, s)k, \quad G \in \mathcal{Y}. \quad (3.22)$$

Applying above arguments to this initial value problem yields

$$u(t) = W^\Delta(t, s + qT_0)u(s + qT_0) = Q(s + qT_0, t)^* u(s + qT_0)$$

for  $s + qT_0 \leq t < s + \min\{2qT_0, T, T(\alpha', \alpha)\}$ . This implies by the first step  $u(t) = Q(s + qT_0, t)^* Q(s, s + qT_0)^* k$  and hence

$$\langle G, u(t) \rangle = \langle Q(s + qT_0, t)G, Q(s, s + qT_0)^* k \rangle,$$

where  $Q(s, s + qT_0)^*$  is considered as a bounded linear operator in  $L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ . Such operator is the adjoint to  $Q(s, s + qT_0) \in L(\mathbb{E}_\alpha, \mathbb{E}_{\alpha'})$ . Moreover, we can show similarly to (3.15) that  $Q(s, s + qT_0)Q(s + qT_0, t)G = Q(s, t)G$  holds. Altogether, this implies

$$\langle Q(s + qT_0, t)G, Q(s, s + qT_0)^* k \rangle = \langle Q(s, t)G, x \rangle = \langle G, Q(s, t)^* k \rangle.$$

Since also  $W^\Delta(s + qT_0, t)k$  satisfies (3.22) for all  $s + qT_0 \leq t < s + qT_0 + T(\alpha', \alpha)$ , we obtain  $W^\Delta(s, t)k = u(t)$ . An iteration of this scheme yields the first assertion. Similar arguments apply to the second assertion.  $\square$

**Remark 3.11.** *The proof shows that  $Q(s, t)^* k = W^\Delta(t, s)k$  holds.*

## Global worsening evolution systems

In this section we briefly state the results adapted to global worsening evolution systems.

**Definition 3.12.** *A (forward) global worsening evolution system associated to  $L^\Delta(t)$  is a family of operators  $W^\Delta(t, s) : \mathbb{B} \longrightarrow \mathbb{B}$  such that there exists  $M > 0$  satisfying:*

1.  *$W^\Delta(t, s) \in L(\mathbb{B}_{\alpha'}, \mathbb{B}_\alpha)$ , whenever  $0 \leq t - s < T(\alpha', \alpha) := \frac{\alpha - \alpha'}{M}$ . Moreover,  $(t, s) \longmapsto W^\Delta(t, s)k \in \mathbb{B}_\alpha$  is continuously differentiable and satisfies (1.3), (1.4) for any  $\alpha' < \alpha$  and  $k \in \mathbb{B}_{\alpha'}$ .*
2. *For all  $0 \leq t - s < T(\alpha', \alpha)$ , let  $r \in (s, t)$  and  $\alpha'' \in (\alpha', \alpha)$  be such that  $t - r < T(\alpha'', \alpha)$  and  $r - s < T(\alpha', \alpha'')$ , then (1.5) holds.*

A global solution to (1.6) is, by definition, a function  $u : \mathbb{R}_+ \longrightarrow \bigcup_{\alpha > \alpha'} \mathbb{B}_\alpha$  such that for all  $T > 0$  there exists  $\alpha > \alpha'$  and  $u|_{[0, T]}$  is a solution to (1.6) in  $\mathbb{B}_\alpha$ . The next statement gives a sufficient condition for such evolution systems.

**Corollary 3.13.** *Suppose that the same conditions as for Theorem 3.1 are satisfied. Let  $\alpha' > \alpha_*$  and suppose that there exists a sequence  $(\alpha_j)_{j \geq 0}$  such that  $\alpha_j < \alpha_{j+1}$ ,  $\alpha_0 = \alpha'$  and*

$$\sum_{j=0}^{\infty} \frac{\alpha_{j+1} - \alpha_j}{M(\alpha_{j+1})} = \infty \tag{3.23}$$

is satisfied. Then for any  $k \in \mathbb{B}_{\alpha'}$  there exists a unique global solution to (1.6) given by  $W^\Delta(t, s)k$ . In particular, if  $M(\alpha)$  is bounded by  $M^* > 0$ , then the assertions of Theorem 3.1 hold for  $T(\alpha', \alpha) = \frac{\alpha - \alpha'}{2eAM^*}$  and  $W^\Delta(t, s)k$  is a global worsening evolution system.

*Proof.* Let  $k \in \mathbb{B}_{\alpha'}$ , then  $W^\Delta(t, s)k$  is the unique solution to (1.6) on  $[s, s + T(\alpha_0, \alpha_1)]$  in  $\mathbb{B}_{\alpha_1}$ . Fix  $q \in (0, 1)$ , then  $W^\Delta(t, s + qT(\alpha_0, \alpha_1))W^\Delta(s + qT(\alpha_0, \alpha_1), s)k$  yields the unique solution on  $[s + qT(\alpha_0, \alpha_1), s + q(T(\alpha_0, \alpha_1) + T(\alpha_1, \alpha_2))]$  in  $\mathbb{B}_{\alpha_2}$ . By iteration we obtain the unique solution on  $[s, s + q(T(\alpha_0, \alpha_1) + \dots + T(\alpha_N, \alpha_{N+1}))]$  in  $\mathbb{B}_{\alpha_N}$  for any  $N \in \mathbb{N}$ . Such iteration yields a global solution since  $\sum_{j=0}^{\infty} T(\alpha_j, \alpha_{j+1}) = \frac{1}{2eA} \sum_{j=0}^{\infty} \frac{\alpha_{j+1} - \alpha_j}{M(\alpha_{j+1})} = \infty$ . For the second assertion consider  $\alpha_j = \alpha' + j$ , then  $\frac{\alpha_{j+1} - \alpha_j}{M(\alpha_{j+1})} \geq \frac{1}{M^*} > 0$  implies (3.23).  $\square$

The next statement states weak uniqueness for global worsening evolution systems. Its proof follows immediately from Theorem 3.10.

**Theorem 3.14.** *Suppose that the conditions of Theorem 3.10 are satisfied and  $M(\alpha)$  is bounded by  $M^*$ . Then the following assertions hold.*

1. *Let  $T > 0$ ,  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$ ,  $s \geq 0$  and let  $(u(t))_{t \geq s} \subset \mathbb{B}_\alpha$  be continuous in  $t$  w.r.t.  $\sigma(\mathbb{B}_\alpha, \mathbb{E}_\alpha)$  such that (3.20) holds. Then  $u(t) = W^\Delta(t, s)k$  holds for all  $s \leq t$ .*
2. *Let  $T > 0$ ,  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$  and  $(v(s))_{s \in (0, T]} \subset \mathbb{B}_\alpha$  be continuous w.r.t.  $\sigma(\mathbb{B}_\alpha, \mathbb{E}_\alpha)$  such that (3.21) holds. Then  $v(s) = Q^\Delta(s, T)k$  holds for all  $0 < s \leq T$ .*

Below we extend the notion of stability to this case.

**Theorem 3.15.** *Suppose that the conditions of Theorem 3.3 are satisfied and  $M(\alpha)$  is bounded by  $M^*$ . Then for any  $n \in \mathbb{N}$  there exist global worsening evolution systems  $W^{\Delta, n}(t, s)$  and  $W^\Delta(t, s)$  with  $T(\alpha', \alpha) = \frac{\alpha - \alpha'}{2eAM^*}$  corresponding to  $(A_n^\Delta(t), B_n^\Delta(t))$  and  $(A^\Delta(t), B^\Delta(t))$  respectively. Moreover, for any  $\alpha' < \alpha$ ,  $k \in \mathbb{B}_{\alpha'}$  and  $T > 0$*

$$W^{\Delta, n}(t, s)k \longrightarrow W^\Delta(t, s)k, \quad n \rightarrow \infty$$

*holds in  $\mathbb{B}_\alpha$  uniformly in  $0 \leq s \leq t \leq T$ , where  $\alpha > 2eAM^* + \alpha'$ .*

## 4 Infinite system of ordinary differential equations

Let  $L = (a_{nk})_{n,k=0}^\infty$  be an infinite matrix having complex-valued entries and  $x = (x_n)_{n=0}^\infty$  be the initial condition. Consider the corresponding system of ordinary differential equations (1.11). Suppose that  $d_n \geq 0$  and for  $b_{nk}, c_{nk} \in \mathbb{C}$  write

$$a_{nk} = -\delta_{nn}d_n + b_{nk} + c_{nk},$$

where  $\delta_{nk}$  is the Kronecker-delta symbol. We assume that the following conditions are satisfied:

(A) There exists  $\alpha_* < \alpha^*$  and for all  $\alpha \in (\alpha_*, \alpha^*)$  there exists  $q(\alpha) \in (0, 1)$  with

$$e^{-\alpha k} \sum_{n=0}^{\infty} |b_{nk}| e^{\alpha n} \leq q(\alpha) d_k, \quad k \in \mathbb{N}_0.$$

(B) We have  $\sup_{n \in \mathbb{N}} d_n e^{-\nu n} < \infty$  for all  $\nu > 0$ .

(C) There exists a continuous increasing function  $M(\alpha) > 0$  such that for all  $\alpha' < \alpha$

$$e^{-\alpha k} \sum_{n=0}^{\infty} |c_{nk}| e^{\alpha' n} \leq \frac{M(\alpha)}{\alpha - \alpha'}, \quad k \in \mathbb{N}_0.$$

Let  $\mathbb{E}_\alpha := \left\{ u = (u_n)_{n=0}^\infty \mid \sum_{n=0}^{\infty} |u_n| e^{\alpha n} < \infty \right\}$  be the Banach space of all complex-valued sequences equipped with the norm  $\|u\|_{\mathbb{E}_\alpha} := \sum_{n=0}^{\infty} |u_n| e^{\alpha n}$ . Denote by  $\mathbb{E} = (\mathbb{E}_\alpha)_{\alpha \in (\alpha_*, \alpha^*)}$  the corresponding scale of Banach spaces. Define linear operators  $A, B, C$  by  $(Au)_n := -d_n u_n$ ,  $(Cu)_n := \sum_{k=0}^{\infty} c_{nk} u_k$  and  $(Bu)_n := \sum_{k=0}^{\infty} b_{nk} u_k$ , where  $n \in \mathbb{N}_0$  and  $u = (u_n)_{n=0}^\infty$  is any sequence for which the sums are absolutely convergent. Then  $(A, \mathcal{D}_\alpha)$  is the generator of an analytic semigroup of contractions, which preserves the cone of positive sequences. It is considered on its maximal domain  $\mathcal{D}_\alpha := \{u \in \mathbb{E}_\alpha \mid (d_n u_n)_{n=0}^\infty \in \mathbb{E}_\alpha\}$ . Moreover, by property (A), [TV06, Theorem 2.2] and [AR91, Theorem 1.1, Theorem 1.2] it follows that  $(A + B, \mathcal{D}_\alpha)$  is the generator of an analytic semigroup  $T_\alpha(t)$  of contractions on  $\mathbb{E}_\alpha$ . It can be shown that  $T_{\alpha'}(t)$  leaves  $\mathbb{E}_\alpha$  invariant and  $T_{\alpha'}(t)|_{\mathbb{E}_\alpha} = T_\alpha(t)$  holds for all  $\alpha' < \alpha$ . It is not difficult to see that  $A, B, C \in L(\mathbb{E})$ . Condition (C) implies that the operator  $C$  is an Ovcyannikov operator. Hence Theorem 3.9 is applicable, which shows that (1.11) has for every  $x \in \mathbb{E}_\alpha$  a unique classical solution  $u(t)$  in  $\mathbb{E}_{\alpha'}$  for  $0 \leq t < \frac{\alpha - \alpha'}{2eM(\alpha)} =: T(\alpha', \alpha)$ .

Let us consider the adjoint equation to (1.11) on a scale of weighted  $\ell^\infty$ -spaces. Namely, let  $L^* = (a_{kn})_{n,k=0}^\infty$  be the adjoint operator. Given an initial condition  $x = (x_n)_{n=0}^\infty$  we want to solve the system of ordinary differential equations

$$\frac{dv_n(t)}{dt} = (L^* v)_n(t) = \sum_{k=0}^{\infty} a_{kn} v_k(t), \quad v_n(0) = x_n, \quad n \in \mathbb{N}_0. \quad (4.1)$$

Denote by  $\mathbb{B} = (\mathbb{B}_\alpha)_{\alpha \in (\alpha_*, \alpha^*)}$  the dual scale of Banach spaces defined by  $\mathbb{B}_\alpha := \mathbb{E}_\alpha^*$ . Then each  $x \in \mathbb{B}_\alpha$  satisfies  $|x_n| \leq \|x\|_\alpha e^{\beta n}$ ,  $n \in \mathbb{N}_0$ . Theorem 3.10 implies the following.

**Theorem 4.1.** *Let  $\alpha' < \alpha$  and  $x \in \mathbb{B}_{\alpha'}$ . Then there exists a unique collection of functions  $(v(t))_{t \in [0, T(\alpha', \alpha))}$  having the following two properties:*

1. For all  $T \in (0, T(\alpha', \alpha))$  we have  $\sup_{t \in [0, T]} \sup_{n \in \mathbb{N}_0} |v_n(t)| e^{-\alpha n} < \infty$  and  $v_n(t)$  is a continuously differentiable complex-valued function for any  $n \in \mathbb{N}_0$ .
2.  $v(t) = (v_n(t))_{n \in \mathbb{N}_0}$  satisfies the system of ordinary differential equations (4.1) component wise.

Moreover,  $[0, T(\alpha', \alpha)) \ni t \longmapsto v(t) \in \mathbb{B}_\alpha$  is continuously differentiable and satisfies (4.1) in  $\mathbb{B}_\alpha$ , where  $\alpha' < \alpha$ .

The next statement establishes continuous dependence on  $a_{nk}$ . It can be used to show that  $v(t)$  can be approximated by a sequence of solutions  $v^N(t)$  to certain truncated equations. One possible truncation method is given by  $a_{nk}^N = 0$  for  $|n - k| > N$ . In such a case only finitely many terms in (1.11) and (4.1) are non-zero. An approximation by finite dimensional systems of differential equations can be achieved by the truncation  $a_{nk}^N = 0$  for all  $n, k > N$ . Such truncation might be of particular interest for numerical simulations.

**Theorem 4.2.** Let  $L_N := (a_{nk}^N)_{n,k=0}^\infty$ , write  $a_{nk}^N = -\delta_{nn}d_n^N + b_{nk}^N + c_{nk}^N$  and suppose that the conditions (A) – (C) are satisfied for  $M(\alpha), q(\alpha)$  independent of  $n$ . Moreover, assume that

$$\sup_{n \in \mathbb{N}_0} e^{-\alpha n} \sum_{k=0}^{\infty} |a_{kn}^N - a_{kn}| e^{\alpha' k} \longrightarrow 0, \quad N \rightarrow \infty$$

holds for all  $\alpha' < \alpha$ . Let  $v(t)$  be the unique solution to (4.1) and  $v_N(t)$  be the unique solution to (4.1) with  $L_N$  instead. Then

$$\sup_{t \in [0, T]} \sup_{n \in \mathbb{N}_0} |v_n^N(t) - v_n(t)| e^{-\alpha n} \longrightarrow 0, \quad N \rightarrow \infty$$

holds for any  $T \in (0, T(\alpha', \alpha))$  and  $\alpha > \alpha' > \alpha_*$ .

It is not difficult to adapt such arguments to systems of Banach-space valued differential equations. Such equations arise naturally from the analysis of spatial birth-and-death processes, cf. [BK13, KK16].

**Example 4.3.** Let  $v_0(t) = v_0(0) = x_n$  and for  $n \geq 1$  consider

$$\begin{aligned} \frac{dv_n(t)}{dt} = & -n(n-1)v_n(t) - mnv_n(t) + \lambda nv_n(t) \\ & + n(n-1)v_{n-1}(t) - nav_{n+1}(t). \end{aligned}$$

Assume that  $m > 0$ ,  $\lambda > 0$  and  $a > 0$ . Then we may choose  $d_n = n(n-1) + mn$ ,  $b_{n-1,n} = (n-1)n$ ,  $b_{kn} = 0$  for  $k \neq n-1$ ,  $c_{nn} = \lambda n$ ,  $c_{n+1,n} = -na$  and finally  $c_{nk} = 0$  for  $k \notin \{n, n+1\}$ . Previous results are applicable for  $\alpha_* = 0$ ,  $\alpha^* = \infty$ ,  $q(\alpha) = e^{-\alpha}$  and  $M(\alpha) = e^{-1}\lambda + ae^{\alpha-1}$ .

## 5 Fokker-Planck equations for ecological models

Let  $\Gamma$  be the space of all locally finite subsets of  $\mathbb{R}^d$ . We endow  $\Gamma$  with the smallest topology such that for any continuous function  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  having compact support  $\Gamma \ni \gamma \mapsto \sum_{x \in \gamma} f(x)$  is continuous. In particular,  $\Gamma$  is a Polish space, cf. [AKR98]. Let  $(\mathbb{R}^d)^n$  be the collection of all  $(x_1, \dots, x_n)$  such that  $x_j \neq x_k$  whenever  $j \neq k$ . The symmetrization  $\text{sym}_n : (\mathbb{R}^d)^n \rightarrow \Gamma_0^{(n)}$ ,  $(x_1, \dots, x_n) \mapsto \{x_1, \dots, x_n\}$  with  $\Gamma_0^{(n)} = \{\eta \subset \mathbb{R}^d \mid |\eta| = n\}$  is bijective and  $\Gamma_0^{(n)}$  is equipped with the euclidean topology from  $(\mathbb{R}^d)^n$ . Let  $\Gamma_0 = \bigsqcup_{n=0}^{\infty} \Gamma_0^{(n)}$  with  $\Gamma_0^{(0)} = \{\emptyset\}$  be the space of all finite subsets of  $\mathbb{R}^d$ . We consider on  $\Gamma_0$  the topology of disjoint units. In such a case  $\Gamma_0$  is a locally compact Polish space. All functions below are assumed to be Borel measurable on  $\Gamma_0$  or  $\Gamma$ , respectively.

Let  $B_{bs}(\Gamma_0)$  be the space of all bounded functions  $G$  such that there exists  $N(G) \in \mathbb{N}$  and a compact  $\Lambda(G) \subset \mathbb{R}^d$  with

$$G(\eta) = 0, \quad |\eta| > N(G) \text{ or } \eta \cap \Lambda^c \neq \emptyset.$$

Let  $\mathcal{FP}(\Gamma) = \left\{ F(\gamma) = \sum_{\eta \in \gamma} G(\eta) \mid G \in B_{bs}(\Gamma_0) \right\}$ , where  $\in$  means that the sum runs over all finite subsets of  $\gamma$ . For each  $F \in \mathcal{FP}(\Gamma)$  there exists  $N(G) \in \mathbb{N}$ ,  $A(G) \geq 0$  and a compact  $\Lambda(G) \subset \mathbb{R}^d$  such that  $F(\gamma) = F(\gamma \cap \Lambda)$  and

$$|F(\gamma)| \leq A(1 + |\gamma \cap \Lambda|)^N, \quad \gamma \in \Gamma,$$

i.e.  $F$  is a polynomially bounded cylinder function. The Lebesgue-Poisson measure  $\lambda$  on  $\Gamma_0$  is defined by  $\lambda = \delta_{\{\emptyset\}} + \sum_{n=1}^{\infty} \frac{1}{n!} (dx)^{(n)}$ , where  $(dx)^{(n)} := (dx)^{\otimes n} \circ \text{sym}_n^{-1}$  is the pullback of the Lebesgue measure on  $\Gamma_0^{(n)}$ , cf. [KK02]. Let  $\alpha \in \mathbb{R}$  and  $\mathcal{K}_\alpha$  be the Banach space of all equivalence classes of functions  $k : \Gamma_0 \rightarrow \mathbb{R}$  equipped with the norm  $\|k\|_{\mathcal{K}_\alpha} = \text{ess sup}_{\eta \in \Gamma_0} |k(\eta)| e^{-\alpha|\eta|}$ . This space can be identified with the dual space to  $\mathcal{L}_\alpha = L^1(\Gamma_0, e^{\alpha|\cdot|} d\lambda)$  with norm  $\|G\|_{\mathcal{L}_\alpha} = \int_{\Gamma_0} |G(\eta)| e^{\alpha|\eta|} d\lambda(\eta)$ , i.e.  $(\mathcal{L}_\alpha)^* \cong \mathcal{K}_\alpha$ . Denote by  $\mathcal{P}$  the space of all Borel probability measures on  $\Gamma$  such that for each  $\mu \in \mathcal{P}$  there exists  $k_\mu \in \bigcup_{\alpha \in \mathbb{R}} \mathcal{K}_\alpha$  with

$$\int_{\Gamma} \sum_{\eta \in \gamma} G(\eta) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) d\lambda(\eta), \quad G \in B_{bs}(\Gamma_0).$$

The function  $k_\mu$  is called correlation function for  $\mu$  and it is well-known from statistical physics. In the last 15 years it became an important tool in the analysis of Fokker-Planck equations related with birth-and-death Markov evolutions in the continuum, cf. [KKM08,

FKO09] and the references therein. Such Markov evolutions are usually described by a Kolmogorov operator  $L$  acting on a proper set of functions  $F : \Gamma \longrightarrow \mathbb{R}$ , e.g.  $F \in \mathcal{FP}(\Gamma)$ . The particular choice

$$(LF)(\gamma) = \sum_{x \in \gamma} \left( m + \sum_{y \in \gamma \setminus x} a^-(x-y) \right) (F(\gamma \setminus x) - F(\gamma)) \\ + \sum_{x \in \gamma} \int_{\mathbb{R}^d} a^+(x-y) (F(\gamma^+ \cup y) - F(\gamma)) dy$$

is known as the spatial ecological model and was considered in [KK16]. For simplicity of notation, we let  $\gamma \setminus x$  and  $\gamma \cup x$  stand for  $\gamma \setminus \{x\}$  and  $\gamma \cup \{x\}$ . The first term describes the death of a plant located at position  $x \in \gamma$ . The death may be independent of other plants, with constant mortality  $m \geq 0$ , but may also be caused by pair interactions. Such interactions are described by the interaction kernel  $a^-(x-y) \geq 0$ . The second term describes the creation of a new plant at position  $y \in \mathbb{R}^d$  which is produced by another plant  $x \in \gamma$ . The distribution and rate of this creation is described by  $a^+(x-y) \geq 0$ . We suppose that  $a^\pm$  are symmetric, bounded, integrable and that there exists  $\vartheta > 0$  and  $b \geq 0$  such that

$$\sum_{x \in \eta} \sum_{y \in \eta \setminus x} (a^-(x-y) - \vartheta a^+(x-y)) \geq -b|\eta|, \quad \eta \in \Gamma_0 \quad (5.1)$$

holds. Above inequality is a balance condition for the birth and death interactions, it was introduced in [KK16]. In the physical literature such condition is known as stability, cf. [Rue70].

We are interested in the 'statistical description' of the dynamics given by  $L$ . Namely given a probability measure  $\mu_0$  on  $\Gamma$ , one seeks for a family of probability measures  $(\mu_t)_{t \geq 0}$  on  $\Gamma$  such that

$$\frac{d}{dt} \int_{\Gamma} F(\gamma) d\mu_t(\gamma) = \int_{\Gamma} (LF)(\gamma) d\mu_t(\gamma), \quad \mu_t|_{t=0} = \mu_0 \quad (5.2)$$

holds for all  $F \in \mathcal{FP}(\Gamma)$ . Equation (5.2) is known as the Fokker-Planck equation.

**Definition 5.1.** *A weak solution to (5.2) is a family of probability measures  $(\mu_t)_{t \geq 0} \subset \mathcal{P}$  such that  $F, LF \in L^1(\Gamma, d\mu_t)$  holds for any  $t \geq 0$ ,  $F \in \mathcal{FP}(\Gamma)$  and (5.2) is satisfied.*

Introduce the combinatorial transformation

$$(KG)(\gamma) := \sum_{\eta \in \gamma} G(\eta), \quad \gamma \in \Gamma, \quad G \in B_{bs}(\Gamma_0).$$

It has an inverse given by  $(K^{-1}F)(\eta) = \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi)$ . Let  $\hat{L} := K^{-1}LK$  and take  $\mu \in \mathcal{P}$  with correlation function  $k_\mu$ . Then  $\int_{\Gamma} KG(\gamma) d\mu(\gamma) = \int_{\Gamma_0} G(\eta) k_\mu(\eta) d\lambda(\eta)$  and  $\int_{\Gamma} LF(\gamma) d\mu(\gamma) = \int_{\Gamma_0} \hat{L}G(\eta) k_\mu(\eta) d\lambda(\eta)$  hold for each  $G \in B_{bs}(\Gamma_0)$ . These equalities can be used to reformulate (5.2) to, see [FKO09],

$$\frac{d}{dt} \int_{\Gamma_0} G(\eta) k_{\mu_t}(\eta) d\lambda(\eta) = \int_{\Gamma_0} \hat{L}G(\eta) k_{\mu_t}(\eta) d\lambda(\eta), \quad G \in B_{bs}(\Gamma_0). \quad (5.3)$$

The operator  $\hat{L}$  is given by  $\hat{L} = \hat{L}_0 + \hat{L}_1$  with

$$\begin{aligned} (\hat{L}_1 G)(\eta) &= - \left( m|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x-y) \right) G(\eta) + \sum_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y) G(\eta \cup y) dy \\ (\hat{L}_0 G)(\eta) &= - \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x-y) G(\eta \setminus x) + \sum_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y) G(\eta \setminus x \cup y) dy. \end{aligned}$$

Moreover, both operators act as bounded linear operators in the scale  $(\mathcal{L}_\alpha)_{\alpha \in \mathbb{R}}$  such that

$$\|\hat{L}_0 G\|_{\mathcal{L}_{\alpha'}} \leq \frac{m}{e(\alpha - \alpha')} + \frac{\|a^-\|_\infty + \|a^+\|_\infty}{4e^2(\alpha - \alpha')^2} \|G\|_{\mathcal{L}_\alpha}$$

and

$$\|\hat{L}_1 G\|_{\mathcal{L}_{\alpha'}} \leq \frac{\langle a^- \rangle e^\alpha + \langle a^+ \rangle}{e(\alpha - \alpha')} \|G\|_{\mathcal{L}_\alpha}$$

hold for all  $G \in \mathcal{L}_\alpha$  and  $\alpha' < \alpha$ .

**Remark 5.2.** Let  $(k_t)_{t \geq 0}$  be a solution to (5.3). Then, in general,  $k_t$  does not need to be the correlation function of a probability measure  $\mu_t$  on  $\Gamma$ , i.e.  $k_t = k_{\mu_t}$ . For such property additional analysis is required, which should be realized for each model separately.

Let  $L^\Delta$  be defined by

$$\int_{\Gamma_0} \hat{L}G(\eta) k(\eta) d\lambda(\eta) = \int_{\Gamma_0} G(\eta) L^\Delta k(\eta) d\lambda(\eta), \quad G \in B_{bs}(\Gamma_0),$$

where  $k \in \bigcup_{\alpha \in \mathbb{R}} \mathcal{K}_\alpha$ . The operator  $L^\Delta$  is given by  $L^\Delta = L_0^\Delta + L_1^\Delta$ , where

$$\begin{aligned} (L_0^\Delta k)(\eta) &= - \left( m|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x-y) \right) k(\eta) + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^+(x-y) k(\eta \setminus x) \\ (L_1^\Delta k)(\eta) &= - \sum_{x \in \eta} \int_{\mathbb{R}^d} a^-(x-y) k(\eta \cup y) dy + \int_{x \in \eta} \int_{\mathbb{R}^d} a^+(x-y) k(\eta \setminus x \cup y) dy. \end{aligned}$$



Both operators act as a bounded linear operators in the scale  $(\mathcal{K}_\alpha)_{\alpha \in \mathbb{R}}$ . The following is due to [KK16].

**Theorem 5.3.** *Let  $\alpha_0 \in \mathbb{R}$  and  $\mu_0$  be a probability measure on  $\Gamma$  having correlation function  $k_0 \in \mathcal{K}_{\alpha_0}$ . Then there exists a unique family  $(k_t)_{t \geq 0} \subset \bigcup_{\alpha \in \mathbb{R}} \mathcal{K}_\alpha$  with the following properties:*

1. *For each  $T > 0$  there exists  $\alpha_T \geq \alpha_0$  such that  $k_t \in \mathcal{K}_{\alpha_T}$ ,  $t \in [0, T)$  and  $(k_t)_{t \in [0, T)}$  is the unique classical solution to*

$$\frac{\partial k_t}{\partial t} = L^\Delta k_t, \quad k_t|_{t=0} = k_0 \quad (5.4)$$

*in  $\mathcal{K}_{\alpha_T}$ .*

2. *There exists a unique family of probability measures  $(\mu_t)_{t \geq 0}$  on  $\Gamma$  such that for any  $t \geq 0$ ,  $k_t$  is the correlation function to  $\mu_t$ .*

Note that existence and uniqueness is only established for classical solutions to (5.4). Although an evolution of states  $(\mu_t)_{t \geq 0} \subset \mathcal{P}$  was constructed, its relation to (5.2) was left open.

**Lemma 5.4.** *Let  $\alpha' < \alpha$  with  $\alpha' > \ln(\vartheta)$  and  $k_0 \in \mathcal{K}_{\alpha'}$  be arbitrary. Then there exists a unique weak solution  $(k_t)_{t \in [0, T(\alpha', \alpha))} \subset \mathcal{K}_\alpha$  to (5.3), where  $T(\alpha', \alpha) = \frac{\langle a^+ \rangle + b + \langle a^- \rangle e^\alpha}{e(\alpha - \alpha')}$ .*

*Proof.* Let  $\hat{L}_{0,b}G := \hat{L}_0G - b|\eta|G$  and for  $\beta > \ln(\vartheta)$  let  $\mathcal{D}_\beta := \{G \in \mathcal{L}_\beta \mid M \cdot G \in \mathcal{L}_\beta\}$ , where  $M(\eta) = m|\eta| + \sum_{x \in \eta} \sum_{y \in \eta \setminus x} a^-(x - y)$ . Condition (5.1) implies that

$$\int_{\Gamma_0} \sum_{x \in \eta} \int_{\mathbb{R}^d} |G(\eta \cup y)| dy e^{\beta|\eta|} d\lambda(\eta) \leq \frac{e^{-\beta}}{\vartheta} \int_{\Gamma_0} M(\eta) |G(\eta)| e^{\beta|\eta|} d\lambda(\eta), \quad 0 \leq G \in \mathcal{D}_\beta.$$

By [TV06] and [AR91] it follows that  $(\hat{L}_{0,b}, \mathcal{D}_\beta)$  is the generator of a positive, analytic semigroup  $(S_\beta(t))_{t \geq 0}$  of contractions on  $\mathcal{L}_\beta$ . It is not difficult to see that  $S_{\beta'}(t)$  leaves  $\mathcal{L}_\beta$  invariant and  $\bar{S}_{\beta'}(t)|_{\mathcal{L}_\beta} = S_\beta(t)$  holds for all  $\ln(\vartheta) < \beta' < \beta$ . Therefore  $(S_\beta(t))_{t \geq 0}$  determines a time-homogeneous evolution system in the scale  $(\mathcal{L}_\beta)_{\beta > \ln(\vartheta)}$ . Let  $\hat{L}_{1,b}G := \hat{L}_1G + b|\eta|G$ , then  $\hat{L} = \hat{L}_{0,b} + \hat{L}_{1,b}$  and for any  $\beta' < \beta$  and  $G \in \mathcal{L}_\beta$

$$\|\hat{L}_{1,b}G\|_{\mathcal{L}_{\beta'}} \leq \frac{\langle a^- \rangle e^\alpha + b + \langle a^+ \rangle}{e(\alpha - \alpha')} \|G\|_{\mathcal{L}_\alpha}.$$

Applying Theorem 3.10 implies the assertion. □

The next theorem shows weak uniqueness for (5.2).

**Theorem 5.5.** *The family  $(\mu_t)_{t \geq 0}$ , constructed in Theorem 5.3, is a weak solution to (5.2). Moreover, let  $(\nu_t)_{t \geq 0} \subset \mathcal{P}$  be another weak solution to (5.2) and suppose that for any  $T > 0$  there exists  $\beta_T \geq \alpha_0$  with  $(k_{\nu_t})_{t \in [0, T]} \subset \mathcal{K}_{\beta_T}$ . Then  $\mu_t = \nu_t$  for all  $t \geq 0$ .*

*Proof.* Let  $(k_{\mu_t})_{t \geq 0}$  be the family of correlation functions corresponding to  $(\mu_t)_{t \geq 0} \subset \mathcal{P}$ . Since it is a unique classical solution to (5.4), it is also a solution to (5.3). One can show that (5.2) and (5.3) are equivalent. Hence  $(\mu_t)_{t \geq 0}$  is a weak solution to (5.2). The second assertion follows from previous lemma by iteration.  $\square$

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## References

- [AKR98] S. Albeverio, Y. Kondratiev, and M. Röckner. Analysis and geometry on configuration spaces. *J. Funct. Anal.*, 154(2):444–500, 1998.
- [AR91] W. Arendt and A. Rhandi. Perturbation of positive semigroups. *Arch. Math. (Basel)*, 56(2):107–119, 1991.
- [BA06] J. Banasiak and L. Arlotti. *Perturbations of positive semigroups with applications*. Springer Monographs in Mathematics. Springer-Verlag London, Ltd., London, 2006.
- [Bea72] R. Beals. Semigroups and abstract gevrey spaces. *J. Funct. Anal.*, 10(3):300–308, 1972.
- [BHP15] R. F. Barostichi, A. A. Himonas, and G. Petronilho. A Cauchy-Kovalevsky theorem for nonlinear and nonlocal equations. In *Analysis and geometry*, volume 127 of *Springer Proc. Math. Stat.*, pages 59–68. Springer, Cham, 2015.
- [BKKK13] C. Berns, Y. Kondratiev, Y. Kozitsky, and O. Kutoviy. Kawasaki dynamics in continuum: micro- and mesoscopic descriptions. *J. Dyn. Diff. Equat.*, 25(4):1027–1056, 2013.
- [BLM06] J. Banasiak, M. Lachowicz, and M. Moszyński. Semigroups for generalized birth-and-death equations in  $l^p$  spaces. *Semigroup Forum*, 73(2):175–193, 2006.
- [EN00] K. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.
- [Fin15] D. Finkelshtein. Around Ovsiyannikov’s method. *Methods Funct. Anal. Topology*, 21(2):134–150, 2015.
- [FK13] M. Friesen and O. Kutoviy. On nonautonomous Markov evolutions in continuum. *Interdisciplinary Studies of Complex Systems*, 2:5–59, 2013.
- [FKKO15] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and M. J. Oliveira. Dynamical Widom-Rowlinson model and its mesoscopic limit. *J. Stat. Phys.*, 158(1):57–86, 2015.
- [FKKZ14] D. Finkelshtein, Y. Kondratiev, O. Kutoviy, and E. Zhizhina. On an aggregation in birth-and-death stochastic dynamics. *Nonlinearity*, 27(6):1105–1133, 2014.
- [FKO09] D. Finkelshtein, Y. Kondratiev, and M. J. Oliveira. Markov evolutions and hierarchical equations in the continuum. I. One-component systems. *J. Evol. Equ.*, 9(2):197–233, 2009.
- [FKO12] D. L. Finkelshtein, Y. G. Kondratiev, and M. J. Oliveira. Glauber dynamics in the continuum via generating functionals evolution. *Complex Anal. Oper. Theory*, 6(4):923–945, 2012.
- [Hen13] H. R. Henríquez. Existence of solutions of the nonautonomous abstract Cauchy problem of second order. *Semigroup Forum*, 87(2):277–297, 2013.
- [Kat70] T. Kato. Linear evolution equations of “hyperbolic” type. *J. Fac. Sci. Univ. Tokyo Sect. I*, 17:241–258, 1970.

- [Kat73] T. Kato. Linear evolution equations of “hyperbolic” type. II. *J. Math. Soc. Japan*, 25:648–666, 1973.
- [KK02] Y. Kondratiev and T. Kuna. Harmonic analysis on configuration space. I. General theory. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 5(2):201–233, 2002.
- [KK16] Y. Kondratiev and Y. Kozitsky. The evolution of states in a spatial population model. *J. Dyn. Diff. Equat.*, 28(1):1–39, 2016.
- [KKM08] Y. Kondratiev, O. Kutoviy, and R. Minlos. On non-equilibrium stochastic dynamics for interacting particle systems in continuum. *J. Funct. Anal.*, 255(1):200–227, 2008.
- [Kol13] V. N. Kolokoltsov. Nonlinear Lévy and nonlinear Feller processes: an analytic introduction. In *Mathematics and life sciences*, volume 1 of *De Gruyter Ser. Math. Life Sci.*, pages 45–69. De Gruyter, Berlin, 2013.
- [Lem10] L. Lemle. Existence and uniqueness for  $C_0$ -semigroups on the dual of a Banach space. *Carpathian J. Math.*, 26(1):67–76, 2010.
- [Ovs74] L. V. Ovsjannikov. The Cauchy problem in a scale of Banach spaces of analytic functions. In *Proceedings of the Symposium on Continuum Mechanics and Related Problems of Analysis (Tbilisi, 1971), Vol. 2 (Russian)*, pages 219–229. Izdat. “Mecniereba”, Tbilisi, 1974.
- [Ovs80] L. V. Ovsjannikov. Abstract form of the Cauchy-Kowalewski theorem and its application. In *Partial differential equations (Proc. Conf., Novosibirsk, 1978) (Russian)*, pages 88–94, 250. “Nauka” Sibirsk. Otdel., Novosibirsk, 1980.
- [Ovs13] L. V. Ovsyannikov. Cauchy problem in a scale of Banach spaces. *Proc. Steklov Inst. Math.*, 281(1):3–11, 2013.
- [Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied mathematical sciences ; 44*. Springer, New York [u.a.], 2., corr. print edition, 1983.
- [Rue70] D. Ruelle. Superstable interactions in classical statistical mechanics. *Comm. Math. Phys.*, 18:127–159, 1970.
- [RZ10] M. Röckner and X. Zhang. Weak uniqueness of fokker–planck equations with degenerate and bounded coefficients. *Comptes Rendus Mathématique*, 348(7):435–438, 2010.
- [Saf95] M. V. Safonov. The abstract Cauchy-Kovalevskaya theorem in a weighted Banach space. *Comm. Pure Appl. Math.*, 48(6):629–637, 1995.
- [TV06] H. R. Thieme and J. Voigt. Stochastic semigroups: their construction by perturbation and approximation. In *Positivity IV—theory and applications*, pages 135–146. Tech. Univ. Dresden, Dresden, 2006.
- [WZ02] L. Wu and Y. Zhang. Existence and uniqueness of  $C_0$ -semigroup in  $L^\infty$ : a new topological approach. *C. R. Math. Acad. Sci. Paris*, 334(8):699–704, 2002.
- [WZ06] L. Wu and Y. Zhang. A new topological approach to the  $L^\infty$ -uniqueness of operators and the  $L^1$ -uniqueness of Fokker-Planck equations. *J. Funct. Anal.*, 241(2):557–610, 2006.